discourse among students, and serve as tools for solving problems. As students use and make connections among contextual, physical, visual, verbal, and symbolic representations, they grow in their appreciation of mathematics as a unified, coherent discipline. The teacher and student actions listed in the table below provide a summary of what teachers and students do in using mathematical representations in teaching and learning mathematics.

Use and connect mathematical representations
Teacher and student actions

What are *teachers* doing?	What are *students* doing?
Selecting tasks that allow students to decide which representations to use in making sense of the problems.	Using multiple forms of representations to make sense of and understand mathematics.
Allocating substantial instructional time for students to use, discuss, and make connections among representations.	Describing and justifying their mathematical understanding and reasoning with drawings, diagrams, and other representations.
Introducing forms of representations that can be useful to students.	Making choices about which forms of representations to use as tools for solving problems.
Asking students to make math drawings or use other visual supports to explain and justify their reasoning.	Sketching diagrams to make sense of problem situations.
Focusing students' attention on the structure or essential features of mathematical ideas that appear, regardless of the representation.	Contextualizing mathematical ideas by connecting them to real-world situations.
Designing ways to elicit and assess students' abilities to use representations meaningfully to solve problems.	Considering the advantages or suitability of using various representations when solving problems.

Facilitate Meaningful Mathematical Discourse

Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Effective mathematics teaching engages students in discourse to advance the mathematical learning of the whole class. Mathematical discourse includes the purposeful exchange of ideas through classroom discussion, as well as through other forms of verbal, visual, and written communication. The discourse in the mathematics classroom gives students opportunities to share ideas and clarify understandings, construct convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives (NCTM 1991, 2000).

Discussion

Discourse that focuses on tasks that promote reasoning and problem solving is a primary mechanism for developing conceptual understanding and meaningful learning of mathematics (Michaels, O'Connor, and Resnick 2008). According to Carpenter, Franke, and Levi (2003, p. 6),

> Students who learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding that is critical to their future success in mathematics and related fields.

Although discourse provides important opportunities for students to learn what mathematics is *and* how one does it, creating a culture of discourse in the mathematics classroom also presents challenges. Teachers must determine how to build on and honor student thinking while ensuring that the mathematical ideas at the heart of the lesson remain prominent in class discussions (Engle and Conant 2002). For example, in orchestrating a class discussion of student approaches to solving a task, the teacher must decide what approaches to share, the order in which they should be shared, and the questions that will help students make connections among the different strategies and the key disciplinary ideas that are driving the lesson. Such discussions can easily become little more than elaborate show-and-tell sessions (Wood and Turner-Vorbeck 2001) in which it is not clear what each solution adds to students' developing understanding or how it advances the mathematical storyline of the lesson. Smith and Stein (2011) describe five practices for effectively using student responses in whole-class discussions:

1. *Anticipating* student responses prior to the lesson
2. *Monitoring* students' work on and engagement with the tasks
3. *Selecting* particular students to present their mathematical work
4. *Sequencing* students' responses in a specific order for discussion
5. *Connecting* different students' responses and connecting the responses to key mathematical ideas

Students must also have opportunities to talk with, respond to, and question one another as part of the discourse community, in ways that support the mathematics learning of all students in the class. Hufferd-Ackles, Fuson, and Sherin (2004) describe a framework for moving toward a classroom community centered on discourse. They examine how teachers and students proceed through levels in shifting from a classroom in which teachers

play the leading role in pursuing student mathematical thinking to one in which they assist students in taking on important roles. The framework describes growth in five components (Hufferd-Ackles, Fuson, and Sherin 2014):

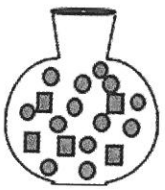
1. How the teacher supports student engagement
2. Who serves as the questioner and what kinds of questions are posed
3. Who provides what kinds of explanations
4. How mathematical representations are used
5. How much responsibility students share for the learning of their peers and themselves

Figure 11 shows a table developed by Hufford-Ackles, Fuson, and Sherin (2014) to describe the levels of classroom discourse through which teachers and their students advance.

Illustration

Mr. Donnelly and his seventh-grade students are studying proportional relationships and their use to solve real-world and mathematical problems. As part of this work, Mr. Donnelly wants his students to be able to identify multiplicative relationships between quantities and recognize three strategies for solving such problems—scaling up, scale factor, and unit rate. He has selected the Candy Jar task, shown in figure 12, for the lesson, since it is aligned with his goals, provides opportunities for high-level reasoning, and offers multiple entry points. Figure 13 shows Mr. Donnelly's lesson on the Candy Jar task.

Suppose you have a new candy jar with the same ratio of Jolly Ranchers (JR) to jawbreakers (JB) as shown in the picture, but it contains 100 Jolly Ranchers.



How many jawbreakers do you have?

Justify your answer.

Note: In the picture, Jolly Ranchers are represented by 5 rectangles, and jawbreakers are shown by 13 circles.

Fig. 12. The Candy Jar task. Adapted from Smith and colleagues (2005).

	Teacher role	Questioning	Explaining mathematical thinking	Mathematical representations	Building student responsibility within the community
Level 0	Teacher is at the front of the room and dominates conversation.	Teacher is only questioner. Questions serve to keep students listening to teacher. Students give short answers and respond to teacher only.	Teacher questions focus on correctness. Students provide short answer-focused responses. Teacher may give answers.	Representations are missing, or teacher shows them to students.	Culture supports students keeping ideas to themselves or just providing answers when asked.
Level 1	Teacher encourages the sharing of math ideas and directs speaker to talk to the class, not to the teacher only.	Teacher questions begin to focus on student thinking and less on answers. Only teacher asks questions.	Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in an explanation. Students provide brief descriptions of their thinking in response to teacher probing.	Students learn to create math drawings to depict their mathematical thinking.	Students believe that their ideas are accepted by the classroom community. They begin to listen to one another supportively and to restate in their own words what another student has said.
Level 2	Teacher facilitates conversation between students, and encourages students to ask questions of one another.	Teacher asks probing questions and facilitates some student-to-student talk. Students ask questions of one another with prompting from teacher.	Teacher probes more deeply to learn about student thinking. Teacher elicits multiple strategies. Students respond to teacher probing and volunteer their thinking. Students begin to defend their answers.	Students label their math drawings so that others are able to follow their mathematical thinking.	Students believe that they are math learners and that their ideas and the ideas of their classmates are important. They listen actively so that they can contribute significantly.
Level 3	Students carry the conversation themselves. Teacher only guides from the periphery of the conversation. Teacher waits for students to clarify thinking of others.	Student-to-student talk is student initiated. Students ask questions and listen to responses. Many questions ask "why" and call for justification. Teacher questions may still guide discourse.	Teacher follows student explanations closely. Teacher asks students to contrast strategies. Students defend and justify their answers with little prompting from the teacher.	Students follow and help shape the descriptions of others' math thinking through math drawings and may suggest edits in others' math drawings.	Students believe that they are math leaders and can help shape the thinking of others. They help shape others' math thinking in supportive, collegial ways and accept the same support from others.

Fig. 11. Levels of classroom discourse. From Hufford-Ackles, Fuson, and Sherin (2014), table 1.

Teacher and student actions

Mathematical discourse among students is central to meaningful learning of mathematics. Teachers carefully prepare and purposefully facilitate discourse, such as whole-class discussions that build on student thinking and guide the learning of the class in a productive disciplinary direction. Students are active members of the discourse community as they explain their reasoning and consider the mathematical explanations and strategies of their classmates. The actions listed in the table below provide some guidance on what teachers and students do as they engage in meaningful discourse in the mathematics classroom.

Facilitate meaningful mathematical discourse **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Engaging students in purposeful sharing of mathematical ideas, reasoning, and approaches, using varied representations. Selecting and sequencing student approaches and solution strategies for whole-class analysis and discussion. Facilitating discourse among students by positioning them as authors of ideas, who explain and defend their approaches. Ensuring progress toward mathematical goals by making explicit connections to student approaches and reasoning.	Presenting and explaining ideas, reasoning, and representations to one another in pair, small-group, and whole-class discourse. Listening carefully to and critiquing the reasoning of peers, using examples to support or counterexamples to refute arguments. Seeking to understand the approaches used by peers by asking clarifying questions, trying out others' strategies, and describing the approaches used by others. Identifying how different approaches to solving a task are the same and how they are different.

Pose Purposeful Questions

Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Effective mathematics teaching relies on questions that encourage students to explain and reflect on their thinking as an essential component of meaningful mathematical discourse. Purposeful questions allow teachers to discern what students know and adapt lessons to meet